The Averaging Trick

representations $\varphi : G \to GL(V)$

$\psi : G \to GL(W)$

linear map $T \in \mathrm{Hom}(V, W)$

Prop: Define:

$$T' := \frac{1}{|G|} \sum_{g \in G} \psi_g \circ T \circ \varphi_{g^{-1}}$$

Then $T' \in \mathrm{Hom}_G(\varphi, \psi)$

Proof: $$\boxed{T' := \frac{1}{|G|} \sum_{g\in G}' \psi_g T \varphi_{g^{-1}}}$$

Show $\forall\, a \in G$, $\quad \psi_a T' = T' \varphi_a$

i.e., $\quad \psi_a T' \varphi_{a^{-1}} = T'$

$$\psi_a T' \varphi_{a^{-1}} = \frac{1}{|G|} \sum_{g\in G}' \psi_a \psi_g T \varphi_{g^{-1}} \varphi_{a^{-1}}$$

$$= \frac{1}{|G|} \sum_{g\in G}' \psi_{ag} T \varphi_{(ag)^{-1}}$$

$$= \frac{1}{|G|} \sum_{h\in G}' \psi_h T \varphi_{h^{-1}} \qquad h = ag$$

$$" T' \quad \checkmark$$

Ex: $T \in \mathrm{Hom}_G(\varphi, \psi) \implies T' = T$

Rem: Uses $|G| < \infty$, and F has characteristic 0

i.e. $\underbrace{1 + \cdots + 1}_{n} \neq 0$; $\quad$ (if $p \mid |G|$, and $F = \mathbb{Z}_p$, trick doesn't work).

Example:

$$\varphi, \psi : \mathbb{Z}_4 \longrightarrow GL_2(\mathbb{C})$$

$$\varphi_{[k]} := A^k, \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\psi_{[k]} := B^k, \qquad B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$T = I$, so

$$T' = \frac{1}{4}\left[I + BA^{-1} + B^2A^{-2} + B^3A^{-3} \right]$$

$$= \frac{1}{4}\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right]$$

$$= \frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$ gives morphism φ to ψ

$$\Rightarrow \quad \varphi \sim \psi$$